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# Magnetic Charge Quantization and Generalized Imprimitivity Systems

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*Abstract*

The quantum mechanical concept of an active translation operation in an external magnetic field is discussed, and an integral version of the kinetic momentum components' commutation relations in terms of a generalized imprimitivity system is formulated. Magnetic charge quantization then follows from a cocyclelike identity in complete analogy with Dirac's original derivation. A generalized system of imprimitivity for the Dirac monopole is explicitly constructed with no strings attached.

## 1. *Introduction*

The Dirac quantization condition for a magnetic charge $g$ was originally derived (Dirac, 1931) on an intuitive basis. Many authors attempted, in a more or less rigorous way, to rederive the Dirac result, but, in our opinion, none of these attempts can be considered as satisfactory. In particular, in the papers of Goldhaber (1965), Hurst (1968), Peres (1958), and Lipkin et al. (1969), rotational invariance has been exploited, while in the original Dirac derivation no such argument was used. The present paper may be considered as an attempt at finding what kind of mathematical structure is hidden behind the Dirac ideas.

In order to describe a charged, spinless particle in an external magnetic field $\mathbf{B}$, we introduce a concept of *generalized imprimitivity system* (GIS) associated with $\mathbf{B}$. Then we show that the Dirac quantization condition is a direct consequence of some analyticity and associativity of multiplication of Hilbert space operators. Some of our formulas are so closely related to those of Dirac's (1931) paper, that it is, in fact, plausible that our approach is a refinement of the original Dirac one; however, our language is different. We are able to prove that quantization occurs for every finite system of magnetic charges, and so has nothing to do with rotational invariance.

The real goal of the present paper is a rigorous construction of a GIS corresponding to $\mathbf{B} = g\mathbf{r}/r^3$ with $g = (\hbar c/2e)k$, where $k$ is an integer. Our approach goes along the lines of Goldhaber (1965) and Lipkin et al. (1969). At the present time we can not give a rigorous and general proof of the uniqueness of our construction.

## 2. *GIS Associated with a Given Field* B

The quantum mechanics of free nonrelativistic elementary systems seems to be well understood now, after the work of Bargmann (1951) and Mackey (see Mackey, 1968, for a clear review). This is not, however, the case with external field problems. When an external field is present and Galilean symmetry is broken, we get out of our depth and there remains nothing but to rely on conventional, well-established recipes. As has been emphasized by Ekstein (1966), it would be desirable to replace these recipes by a set of clear and cogent "first principles" one can use as a firm basis when the symmetry breaks down. Ekstein (1966, 1967, 1969) looks for such principles with the aim of understanding the origin of canonical commutation relations for systems in external fields. Indeed, it is one of the fundamental recipes that tells us to start always with the variables $\{q_i, p_j\}$ satisfying canonical commutation relations. However, the meaning of the canonical momenta is rather obscure. While in classical mechanics everything can be expressed in terms of fundamental observables, coordinates and velocities, and the canonical momentum plays only the role of an auxiliary variable, in quantum theory the situation is much more intricate. In a magnetic field the velocity components cease to commute and can no longer be easily used for identification of states. Moreover, the velocity commutation relations are not universal, being dependent on the field. On the other hand, canonical momenta have been successfully used for description of scattering amplitudes and asymptotic states. Presumably, it is just for these reasons that Ekstein (1966) attempts to save canonical momentum and give to it an operational meaning.

To our knowledge, nobody has attempted to measure a canonical momentum in the framework of classical mechanics, and, after all, one can hardly imagine such a measurement without a prior specification of, otherwise arbitrary, electromagnetic potentials. What is measured by a ballistic pendulum is not a canonical, as suggested by Ekstein (1966), but rather the kinetic momentum. We see no reason why in quantum theory the situation should be different. The only quantum effect will manifest itself in limitations (à la Heisenberg) related to the kinetic momentum commutation relations, and, as far as arbitrariness of gauge transformations is maintained, one can hardly believe that canonical momenta are genuine observables.

Consider a nonrelativistic, spinless particle in an external magnetic field **B**. According to the conventional, canonical procedure let A be a vector potential and $\{q, p\}$ an irreducible representation of canonical commutation relations. The Hamiltonian $H$ is given by

$$H = \pi^2/2m$$

where

$$\boldsymbol{\pi} = \mathbf{p} - (e/c)A$$

are the kinetic momentum operators:

$$\boldsymbol{\pi} = m\dot{\mathbf{q}}$$

Let us take $\mathbf{x} := \mathbf{q}$ and $\boldsymbol{\pi}$ as fundamental observables. They satisfy a kind of generalized canonical commutation relations, namely

(i) $[x_i, x_j] = 0$

(ii) $[x_i, \pi_j] = i\hbar\delta_{ij}$

(iii) $[\pi_i, \pi_j] = i(e\hbar/c)\epsilon_{ijk}B_k$

The third relation can be easily understood on physical grounds once one analyzes the problem of preparing quantum states in a magnetic field. Let us suppose we have an apparatus that produces particles with a small momentum dispersion in the absence of external fields. In order not to disturb the action of the apparatus one may, for example, switch on the field **B** immediately after a particle leaves the machine. However, during the time interval in which **B** grows from zero to its final value, an electric field is necessarily produced, and it is easy to see that the spatial extension of the apparatus makes a disturbance of different components of the kinetic momentum unavoidable. The result can be seen to be in full agreement with uncertainty relations following from (iii).

Once the relations (i)–(iii) are physically understood, we can try to find a satisfactory mathematical theory to handle them. In case of $\mathbf{B} = 0$, one can put the canonical commutation relations either in a Weyl form or in some form of imprimitivity system. In the latter approach the basic concepts have a clear physical meaning: the spectral measure on the configuration space and the unitary representation of the translation group. Let us therefore introduce them in the general case of $\mathbf{B} \neq \mathbf{0}$. Let $E: S \mapsto E(S)$ be a spectral measure on $\mathbf{R}^3$ such that

$$q_i = \int x_i \, dE(\mathbf{x})$$

and let

$$U(\mathbf{a}) = \exp\,[(-i/\hbar)\boldsymbol{\pi}\cdot\mathbf{a}]$$

Then (ii) is equivalent to

$$U(\mathbf{a})E(S)U(\mathbf{a})^* = E(S + \mathbf{a}) \qquad (2.1)$$

and (iii) takes a form

$$U(\mathbf{b})U(\mathbf{a}) = U(\mathbf{a} + \mathbf{b})M_B(\mathbf{a}, \mathbf{b}) \qquad (2.2)$$

where $M_B$ is a unitary multiplier commuting with $E$ and functionally dependent on **B**. Before we start a mathematical discussion of these relations let us try to understand the physical meaning of the unitary operators $U(\mathbf{a})$. Keeping in mind the obscurity of the notion of translational symmetry in an external field, one should be very careful when trying to give $U(\mathbf{a})$ the meaning of an active transformation. Common sense seems to suggest that to every state $\psi$

there should correspond a definite state $\psi_a$ that differs from $\psi$ in the position probability distribution only, and in nothing else. However, from the uncertainty relations corresponding to (iii), it follows that in an inhomogeneous field the velocity probability distributions of $\psi$ and $\psi_a$ must be, in general, necessarily different, and so common sense is misleading in this case. Nevertheless, one is still inclined to think about an active translation of a state-preparing apparatus and look for a corresponding mathematical counterpart. Let us analyze this problem in some detail. Classically, during a rigid motion of an apparatus each of its points encircles some path $l = [z(t) : 0 \leqslant t \leqslant T]$. The particle in question will also follow such a path, and so during the motion the Lorentz force will change its kinetic momentum by

$$\Delta\boldsymbol{\pi} = \frac{e}{c}\int_0^T \dot{\mathbf{z}} \times \mathbf{B}\, dt = -\left(\frac{e}{c}\right)\int_l \mathbf{B} \times d\mathbf{z}$$

Thus, we expect, in the quantum case, that the resulting state $\psi_l$ will satisfy (in the limit $T \to 0$, admissible in the nonrelativistic context)

$$(\psi_l, E(S)\psi_l) = (\psi, E(S - \mathbf{a})\psi)$$

and

$$(\psi_l, \boldsymbol{\pi}\psi_l) = \left(\psi, \boldsymbol{\pi} - \frac{e}{c}\int \mathbf{B} \times d\mathbf{z}\; \psi\right)$$

where

$$\mathbf{a} = \mathbf{z}(T)$$

In case of $l$ being a straight line: $\mathbf{z} = \mathbf{x} + (t/T)\mathbf{a}$, we get

$$(\psi_a, \boldsymbol{\pi}\psi_a) = \left(\psi, \left[\boldsymbol{\pi} + \frac{e}{c}\,\mathbf{a} \cdot \int_0^1 \mathbf{B}(\mathbf{x} + s\mathbf{a})\, ds\right]\psi\right)$$

On the other hand, it follows easily from (iii) that

$$U(\mathbf{a})^*\boldsymbol{\pi}U(\mathbf{a}) = \boldsymbol{\pi} + \frac{e}{c}\,\mathbf{a} \cdot \int_0^1 \mathbf{B}(\mathbf{x} + s\mathbf{a})\, ds$$

Thus $U(\mathbf{a})$ can be thought of as implementing an active translation of states along a straight line. A general translation $U(l)$ can be defined by an approximation of $l$ by broken lines $\{\mathbf{a}_1 \ldots, \mathbf{a}_n\}$ and an approximation of $U(l)$ by $U(\mathbf{a}_n) \cdots U(\mathbf{a}_1)$. It should be noticed that our paths are free paths, i.e., paths in the translation group rather than in configuration space.

Having in mind the above physical interpretation of $U(\mathbf{a})$, one can start calculating the multiplier $M_B$. Assuming analyticity enough to make all series and differentiations convergent (i.e., a common dense set of analytic vectors for $\pi_i$-s, or something like that), one arrives at the following expression:

$$M_B(\mathbf{a}, \mathbf{b}) = \int m_B(\mathbf{a}, \mathbf{b}; \mathbf{x})\, dE\,(\mathbf{x}) \tag{2.3}$$

where

$$m_B(\mathbf{a}, \mathbf{b}; \mathbf{x}) = \exp\,[i\Phi_B(\mathbf{a}, \mathbf{b}; \mathbf{x})] \tag{2.4}$$

and

$$\Phi_B(\mathbf{a}, \mathbf{b}; \mathbf{x}) = \frac{e}{\hbar c} \iint_{\mathbf{x} + s(\mathbf{a}, \mathbf{b})} \mathbf{B} \cdot \mathbf{n}(\mathbf{a}, \mathbf{b})\, d\Sigma \tag{2.5}$$

where we use

$$s(\mathbf{a}, \mathbf{b}) = \{t\mathbf{a} + t'(\mathbf{a} + \mathbf{b}) : t, t' \geqslant 0, t + t' \leqslant 1\}$$

A general covariant version has the following form:

$$U(l_2)U(l_1) = U(l_2 \circ l_1)$$

and

$$U(l) = \exp\left(\frac{ie}{\hbar c} \iint_{\mathbf{x}+s(l)} \mathbf{B} \cdot \mathbf{n}\, d\Sigma\right) \tag{2.6}$$

for a loop $l$.

The above considerations lead us to the following definition:

*Definition* 2.1. Let $\mathbf{B}$: $\mathbf{R}^3 \to \mathbf{R}^3$ be a real vector field on $\mathbf{R}^3$ such that for every pair $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3$ the function $\mathbf{B} \cdot \mathbf{n}$ is integrable over $\mathbf{x} + s(\mathbf{a}, \mathbf{b})$ for almost all $\mathbf{x}$, where $s(\mathbf{a}, \mathbf{b})$ is given by (2.6). Let $\Phi_B$ and $m_B$ be as in (2.3)-(2.5) and $\Phi_B = 0$ for $\mathbf{a}$ and $\mathbf{b}$ collinear. A generalized imprimitivity system associated with $\mathbf{B}$ is a triple $(\mathscr{H}, E, U)$, whree $E$ is a spectral measure on $\mathbf{R}^3$ in the Hilbert space $\mathscr{H}$ and $\mathbf{a} \mapsto U(\mathbf{a})$ is a Borel function from $\mathbf{R}^3$ into the unitary group of $\mathscr{H}$ such that

$$(p) \quad U(\mathbf{a})E(S)U(\mathbf{a})^* = E(S + \mathbf{a})$$

$$(pp) \quad U(\mathbf{b})U(\mathbf{a}) = U(\mathbf{b} + \mathbf{a})M_B(\mathbf{a}, \mathbf{b})$$

where

$$(ppp) \quad M_B(\mathbf{a}, \mathbf{b}) = \int m_B(\mathbf{a}, \mathbf{b}; \mathbf{x})\, dE\,(\mathbf{x})$$

*Remark 1.* Since $M_B(\mathbf{a}, -\mathbf{a}) = I$ it follows that $U(\mathbf{a})^* = U(-\mathbf{a})$. Moreover, for each $\mathbf{a}$, $s \mapsto U(s\mathbf{a})$ is a one-parameter Borel group of unitary operators, and so the self-adjoint generators $\pi(\mathbf{a})\!: = i\hbar dU(s\mathbf{a})/ds\,|_{s=0}$ exist. It is easy to see that, on an appropriate domain, (iii) is satisfied.

*Remark 2.* For a constant $\mathbf{B}$ we simply get

$$U(\mathbf{b})U(\mathbf{a}) = \exp\left\{-\left(\frac{ie}{2\hbar c}\right) \mathbf{B} \cdot [\mathbf{a} \times \mathbf{b}]\right\} U(\mathbf{a} + \mathbf{b})$$

i.e., a usual projective representation of the additive group of $\mathbf{R}^3$. Every multiplier is equivalent to the above one (see Bargmann, 1951; also Bacry et al., 1970.

*Lemma* 2.1. Let $(\mathscr{H}, E, U)$ be as GIS associated with **B**. Then $E(S) = 0$ if and only if $S$ is of Lebesgue measure zero.

*Proof.* The proof follows immediately from spectral multiplicity theory (see Halmos, 1957; also Varadarajan, 1970), $(pp)$, and uniqueness of the translation invariant Borel measure on $\mathbf{R}^3$. □

*Theorem* 2.1. A necessary condition for a GIS associated with **B** to exist is that, for every triple of linearly independent vectors $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$,

$$\Phi_B(\mathrm{a}, \mathrm{b}, \mathrm{c}; \mathrm{x}) = 2\pi k(\mathrm{a}, \mathrm{b}, \mathrm{c}; \mathrm{x})$$

x—almost everywhere, where $\Phi_B$ stands for the integral of the two-form **B** over the boundary of the three-simplex $\mathrm{x} + s(\mathrm{a}, \mathrm{b}, \mathrm{c})$, and

$$s(\mathrm{a}, \mathrm{b}, \mathrm{c}) = \{t_1\mathrm{a} + t_2(\mathrm{a} + \mathrm{b}) + t_3(\mathrm{a} + \mathrm{b} + \mathrm{c}): \\ t_i \geqslant 0, t_1 + t_2 + t_3 \leqslant 1\}$$

with the exterior orientation.

*Proof.* With $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ as above let $L: = U(\mathrm{a} + \mathrm{b} + \mathrm{c})^* U(\mathrm{c})U(\mathrm{b})U(\mathrm{a})$. Since $U(\mathrm{b})U(\mathrm{a}) = U(\mathrm{a} + \mathrm{b})M(\mathrm{a}, \mathrm{b})$ and $U(\mathrm{c})U(\mathrm{a} + \mathrm{b}) = U(\mathrm{a} + \mathrm{b} + \mathrm{c})M(\mathrm{a} + \mathrm{b}, \mathrm{c})$, it follows that $L = M(\mathrm{a} + \mathrm{b}, \mathrm{c})M(\mathrm{a}, \mathrm{b})$. On the other hand, since

$$L^* = U(-\mathrm{a})U(-\mathrm{b})U(-\mathrm{c})U(\mathrm{a} + \mathrm{b} + \mathrm{c}), \quad U(-\mathrm{b})U(-\mathrm{c}) = U(-\mathrm{b}-\mathrm{c})M(-\mathrm{c}, -\mathrm{b})$$

and

$$U(-\mathrm{a})U(-\mathrm{b}-\mathrm{c}) = U(-\mathrm{a}-\mathrm{b}-\mathrm{c})M(-\mathrm{b}-\mathrm{c}, -\mathrm{a}),$$

it follows that

$$L^* = U(\mathrm{a} + \mathrm{b} + \mathrm{c})^* M(-\mathrm{b}-\mathrm{c}, -\mathrm{a})M(-\mathrm{c}, -\mathrm{b})U(\mathrm{a} + \mathrm{b} + \mathrm{c})$$

Now, $LL^* = I$ leads to

$$M(\mathrm{a} + \mathrm{b}, \mathrm{c})M(\mathrm{a}, \mathrm{b})U(\mathrm{a} + \mathrm{b} + \mathrm{c})^* M(-\mathrm{b}-\mathrm{c}, -\mathrm{a})M(-\mathrm{c}, -\mathrm{b})U(\mathrm{a} + \mathrm{b} + \mathrm{c}) = I$$

and it follows from Definition 2.1 $(p)$, and $(ppp)$ that

$$m(\mathrm{a} + \mathrm{b}, \mathrm{c}; \mathrm{x})m(\mathrm{a}, \mathrm{b}; \mathrm{x})m(-\mathrm{b}-\mathrm{c}, -\mathrm{a}; \mathrm{x} + \mathrm{a} + \mathrm{b} + \mathrm{c})m(-\mathrm{c}, -\mathrm{b}, \mathrm{x} + \mathrm{a} + \mathrm{b} + \mathrm{c}) = 1$$

x—almost everywhere, owing to Lemma 2.1. The statement in Theorem 2.1. follows now from (2.4) and (2.5). □

*Corollary.* Let $\{\mathrm{a}_n\}$ be a sequence of vectors such that $\Sigma 1/|a_n|^2$ is convergent, and let $\{k_n\}$ be a sequence of real numbers such that $\Sigma k_n/|a_n|^2$ is convergent. Let

$$\mathbf{B}(\mathrm{x}): = \hbar c/2e \; \Sigma k_n(\mathrm{x} - \mathrm{a}_n)/|\mathrm{x} - \mathrm{a}_n|^3$$

A necessary condition for a GIS associated with **B** to exist is that $k_n$ is an integer for each $n$.

*Proof.* Since $\{\mathbf{a}_n\}$ has no limit points in $\mathbf{R}^3$, it follows that for each simplex $s(\mathbf{a}, \mathbf{b}, \mathbf{c})$ there is only a finite number of $\mathbf{a}_n$'s contained in it. The statement follows then from Theorem 2.1.□

The field $\mathbf{B} = g\mathbf{x}/|\mathbf{x}|^3$ describes a magnetic charge $g$ placed at the origin. We have seen that quantization of magnetic charge is a direct consequence of associativity and has nothing to do with the rotation group. For a value of an elementary charge we get $g = \hbar c/2e$, as in the paper of Dirac (1931). In Sec. 3 it will be shown that $g = \hbar c/2e$ is also a sufficient condition for the existence of a GIS associated with $\mathbf{B} = g\mathbf{x}/|\mathbf{x}|^3$.

## 3. *GIS for the Dirac Monopole*

In this section we shall describe a generalized system of imprimitivity corresponding to the elementary magnetic charge placed at the origin. In this case, owing to rotational symmetry, one can expect that there should be a unitary representation of $SU(2)$, $A \mapsto U(A)$ such that $U(A)E(S)U(A)^* = E(R_A S)$ and $U(A)U(\mathbf{a})U(A)^* = U(R_A \mathbf{a})$. We shall see that this is indeed the case. Our construction may be viewed as a refinement of the methods of Goldhaber (1965) and Lipkin et al. (1969). On the other hand, there is a formal similarity between the commutation relations (iii) (Sec. 2), for $\mathbf{B}$ as above, and commutation relations for the components of the photon position operator with $g$ corresponding to helicity (see Schwinger, 1970, Chap. 1., Sec. 3). This observation, together with the explicit construction of photon position operator in Jadczyk and Jancewicz (1973) may be considered as a motivation of the foregoing construction. Finally, it is worthwhile noting that we deal here with a manifestation of the general phenomenon that a projective representation comes from a projection of a true representation (Jadczyk, 1973).

*Construction.* Let $K: = \mathbf{C}^{2s+1}$ and $H: = L^2(\mathbf{R}^3, K, d^3x)$. Let $\mathbf{S} = (S_1, S_2, S_3)$ be a triple of self-adjoint generators of an irreducible unitary representation $W: A \mapsto W(A)$ of $SU(2)$ in $K$. Thus $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$ and $S^2 = \hbar^2 s(s+1)$.

For a bounded Borel function $w: \mathbf{R}^3 \to L(K)$ let $\tilde{w}$ stands for a unique bounded operator in $H$ such that $(\tilde{w}f)(\mathbf{x}) = w(\mathbf{x})f(\mathbf{x})$, x—almost everywhere. We note that $\widetilde{wv} = \tilde{w} \cdot \tilde{v}$ and $(w^*) = (\tilde{w})^*$. In particular, if $w(\mathbf{x})$ is almost everywhere self-adjoint (unitary), then $\tilde{w}$ is self-adjoint (unitary.

For every $\mathbf{x}$, $\mathbf{x} \neq \mathbf{0}$ let

$$g(\mathbf{x}): = (-c/e)S \cdot \mathbf{x}/|\mathbf{x}| \tag{3.1}$$

It is clear that $g: (R^3 - \mathbf{0}) \to L(K)$ is bounded and self-adjoint. Let $G: = \tilde{g}$. Then $G$ is a bounded, self-adjoint operator in $H$. It is easy to see that $G$ is unitarily equivalent to $(-c/e)S_3$ ( we do not distinguish between a constant function and its value), and so the spectrum of $G$ consists of the $2s + 1$ points: $g = -m\hbar c/e$, $m = (-s, \ldots, s-1, s)$. We call $G$ a magnetic charge operator. Under a hypothesis of a superselection rule for the magnetic charge, every observable should commute with $G$. Clearly, in order to describe the Dirac monopole, it is enough to take the case of $s = 1/2$. The Hilbert space $H$ is then a direct sum of two subspaces corresponding to the opposite signs of the charge $g = \hbar c/2e$.

We proceed to define a GIS in $H$. The spectral measure $E$ is defined in a canonical way: given a Borel set $S \subset \mathbf{R}^3$, let $I_S$ be the characteristic function of $S$; then

$$[E(S)f](\mathrm{x}) := I_S(\mathrm{x})f(\mathrm{x}) \tag{3.2}$$

Clearly, for each bounded function $w: \mathbf{R}^3 \to L(K)$, the operator $\tilde{w}$ commutes with $E(\ )$. In particular, if for each $A \in SU(2)$ we define

$$[U(A)f](\mathrm{x}) := [V(R_A)W(A)f](\mathrm{x}) \tag{3.3}$$

where

$$[V(R)f](\mathrm{x}) := f(R^{-1}\mathrm{x}) \tag{3.4}$$

then

$$U(A)E(S)U(A)^{-1} = E(R_A S) \tag{3.5}$$

and $A \mapsto U(A)$ is a continuous unitary representation of $SU(2)$ in $H$. We also observe that $E$ and $U(A)$ commute with the magnetic charge operator $G$. The generators $J_i$ of the representation $A \mapsto U(A)$, as well as the coordinates

$$q_i = \int \mathrm{x}_i \, dE(\mathrm{x})$$

thus remain self-adjoint when restricted to a subspace with a definite value of $G$.

To define the map $\mathbf{a} \mapsto U(\mathbf{a})$, let $\mathbf{a}$ be a fixed nonzero vector and $\mathbf{x} \neq r\mathbf{a}$ for all $r \in \mathbf{R}$. Let

$$\begin{aligned} w(\mathrm{a}; \mathrm{x}) := 2|\mathrm{x}|\,|\mathrm{x} + \mathrm{a}|^{-1/2} \{(|\mathrm{x}|\,|\mathrm{x} + \mathrm{a}| + x^2 + \mathrm{a} \cdot \mathrm{x})^{1/2} \\ + i(\boldsymbol{\sigma} \cdot [\mathrm{a} \times \mathrm{x}]/|\mathrm{a} \times \mathrm{x}|)(|\mathrm{x}|\,|\mathrm{x} + \mathrm{a}| - x^2 + \mathrm{a} \cdot \mathrm{x})^{1/2}\} \end{aligned} \tag{3.6}$$

where $\sigma_i$ are the Pauli matrices. Clearly, $\mathrm{x} \mapsto w(\mathrm{a}; \mathrm{x})$ is almost everywhere defined, and a unitary function from $\mathbf{R}^3$ into $L(K)$ (we specify $K$ to be $\mathbf{C}^2$ from this time on; a general case can be treated in a similar way). Let

$$\begin{aligned} W(\mathrm{a}) &:= w(\mathrm{a}; \cdot)\tilde{}\qquad (\mathrm{a} \neq 0) \\ W(0) &:= I \end{aligned} \tag{3.7}$$

From the very construction, $\mathbf{a} \mapsto W(\mathbf{a})$ is a Borel function with values in the unitary group of $H$, and $W(\mathbf{a})$ commutes with the spectral measure $E$ for all $\mathbf{a}$. Let

$$U(\mathrm{a}) := V(\mathrm{a})W(\mathrm{a}) \tag{3.8}$$

with

$$[V(\mathrm{a})f](\mathrm{x}) := f(\mathrm{x} - \mathrm{a}) \tag{3.9}$$

Then $\mathbf{a} \mapsto U(\mathrm{a})$ is a Borel function and

$$U(\mathrm{a})E(S)U(\mathrm{a})^{-1} = E(S + \mathrm{a}) \tag{3.10}$$

*Lemma 3.1.* For all $\mathbf{a}$ the relation $U(\mathrm{a})GU(\mathrm{a})^{-1} = G$ holds.

*Proof.* It can be verified that

$$w(\mathrm{a}; \mathrm{x})g(\mathrm{x})w(\mathrm{a}; \mathrm{x})^* = g(\mathrm{x} + \mathrm{a})$$

for almost all $\mathbf{x}$. Therefore, by (3.7)-(3.9) we have

$$W(\mathrm{a})GW(\mathrm{a})^* = V(\mathrm{a})^*GV(\mathrm{a})$$

which is equivalent to the statement of the Lemma. □

*Lemma* 3.2. For each a, the map $t \mapsto U(t\mathrm{a})$ is a one-parameter Borel group of unitary operators (and so is strongly continuous).

*Proof.* By a direct inspection it can be seen that a cocyclelike relation

$$w(t\mathrm{a}; \mathrm{x} + s\mathrm{a})w(s\mathrm{a}; \mathrm{x}) = w((s + t)\mathrm{a}; \mathrm{x})$$

holds for all $\mathbf{x}$ for which the both sides are defined, i.e., almost everywhere. This is enough to imply

$$U(t\mathrm{a})U(s\mathrm{a}) = U((t + s)\mathrm{a}) \ \square$$

*Lemma* 3.3. For all $A \in SU(2)$ and $\mathrm{a} \in \mathbf{R}^3$, the relation

$$U(A)U(\mathrm{a})U(A)^* = U(R_A \mathrm{a})$$

holds.

*Proof.* We have

$$U(A)U(\mathrm{a})U(A)^* = V(R_A)W(A)V(\mathrm{a})W(\mathrm{a})W(A)^*V(R_A)^*$$
$$= V(R_A)V(\mathrm{a})V(R_A)^*V(R_A)W(A)W(\mathrm{a})W(A)^*V(R_A)^* = V(R_A\mathrm{a})W(R_A\mathrm{a})$$
$$= U(R_A\mathrm{a})$$

where use has been made of (3.3)-(3.9). □

It follows from Lemma 3.2 that there exist self-adjoint generators $\pi(\mathrm{a}) := i\hbar dU(t\mathrm{a})/dt|_{t=0}$. On an appropriate domain they are given by

$$\pi(\mathrm{a}) = \mathbf{p} \cdot \mathrm{a} + ([\mathrm{s} \times \mathrm{x}] \cdot \mathrm{a})/\mathrm{x}^2$$

where $p_i$ are the generators of $\mathrm{a} \mapsto V(\mathrm{a})$. According to Lemma 3.1 the kinetic momenta $\pi_i$ are observables, and according to Lemma 3.3 they transform as a vector under rotations. Clearly, by (3.10), they satisfy $[q_i, \pi_j] = i\hbar\delta_{ij}$, and their commutator, when calculated, is given by

$$\boldsymbol{\pi} \times \boldsymbol{\pi} = i(e\hbar/c)\mathbf{B}$$

where

$$\mathbf{B} := G\mathbf{x}/|\mathbf{x}|^3$$

is the magnetic field operator corresponding to the Dirac monopole.

*Remark.* Clearly, the absence of strings and a complete covariance of the above approach is a consequence of working in the Hilbert space $H$ which contains charges of opposite signs. There is no manifest covariance in a helicity representation of photon states. A similar phenomenon occurs in our case.

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